UNIFORM ASYMPTOTIC STABILITY OF SYSTEMS OF DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER IN THE DERIVATIVE TERMS

(RAVNOMERNAIA ASIMPTOTICHESKAIA USTOICHIVOST' SISTEM DIFFERENTSIAL'NYKH URAVNENII S MALYM PARAMETROM PRI PROIZBODNYKH)

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A.I. KLIMUSHCHEV and N.N. KRASOVSKII
(Sverdlovsk)

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In this paper, under the assumption of uniform asymptotic stability of the degenerate first approximation system and the asymptotic stability of a certain auxiliary system, it is shown that a certain class of systems of differential equations, containing a small parameter among the derivative terms, is asymptotically stable.

The method of Liapunov functions [1] is employed. It should be remarked that the possibility of applying the classical methods of the theory of stability in dealing with systems containing small parameters in the derivative terms had already been pointed out by Chetaev [2]. Also, as is well known, systems with a small parameter in the derivative terms have been studied in great detail by A.N. Tikhonov and his coworkers, and I.S. Gradshtein.

The theorem proved in this paper for the linear case is a generalization of the theorem proved by Razumikhin [3] for systems in which the small parameter occurs in only one equation of the system.

1. Linear systems. Consider first the linear system of differential equations

$$\begin{aligned} \frac{dx_i}{dt} &= \sum_{l=1}^{m} a_{il}(t)x_l + \sum_{s=1}^{n} b_{is}(t)y_s + f_i(t) \qquad (i = 1, \ldots, m) \\ \mu\frac{dy_j}{dt} &= \sum_{l=1}^{m} c_{jl}(t)x_l + \sum_{s=1}^{n} d_{js}(t)y_s \qquad (j = 1, \ldots, n) \end{aligned} \tag{1.1}$$

where μ is a small positive parameter. We shall consider the stability of the solutions of this system which are defined by the initial conditions

$$x_{i0} = g_{i0} \quad (i = 1, \ldots, m), \qquad y_{j0} = p_{j0} \quad (j = 1, \ldots, n) \qquad \text{for } t = t_0 \quad (1.2)$$

These solutions will be denoted as follows:

$$x_i = x_i(t, \mu), \qquad y_j = y_j(t, \mu) \qquad (1.3)$$

Let us suppose that the coefficients $a_{il}(t)$, $b_{is}(t)$, $c_{jl}(t)$, $d_{js}(t)$ and $f_i(t)$ of the system (1.1) are continuous and bounded functions of the variable t, having continuous bounded derivatives for $t_0 \leqslant t < \infty$. Further, we shall employ the condition

$$\begin{vmatrix} d_{11}(t) & d_{12}(t) & \ldots & d_{1n}(t) \\ d_{21}(t) & d_{22}(t) & \ldots & d_{2n}(t) \\ \ldots & \ldots & \ldots & \ldots \\ d_{n1}(t) & d_{n2}(t) & \ldots & d_{nn}(t) \end{vmatrix} > \beta > 0 \quad (\beta - \text{a certain number}) \qquad (1.4)$$

For $\mu = 0$ the degenerate system corresponding to the system (1.1) has the form

$$\frac{dx_i}{dt} = \sum_{l=1}^{m} a_{il}(t)\, x_l + \sum_{s=1}^{n} b_{is}(t)\, y_s + f_i(t), \qquad \sum_{l=1}^{m} c_{jl}(t)\, x_l + \sum_{s=1}^{n} d_{js}(t)\, y_s = 0 \qquad (1.5)$$

and the solutions of this degenerate system (1.5) which satisfy the initial conditions $x_{i0} = g_{i0}$ will be denoted by

$$x_i = x_i(t), \qquad y_j = y_j(t) \qquad (1.6)$$

Let us solve the system of n algebraic equations

$$\sum_{l=1}^{m} c_{jl}(t)\, x_l + \sum_{s=1}^{n} d_{js}(t)\, y_s = 0 \qquad (s = 1, \ldots, n)$$

with respect to $y_1, \ldots, y_n$; and let its solution be

$$y_s = \sum_{i=1}^{m} \lambda_{si}(t)\, x_i \qquad (s = 1, \ldots, n) \qquad (1.7)$$

where the functions $\lambda_{si}(t)$ are continuous bounded functions of t.

The substitution of Expressions (1.7) for the y's into the first m equations of the degenerate system (1.5) leads to the following system of m linear differential equations with variable coefficients:

$$\frac{dx_i}{dt} = \sum_{k=1}^{m} r_{ik}(t)\, x_k + f_i(t) \qquad (i = 1, \ldots, m) \tag{1.8}$$

where the functions $r_{ik}(t)$ are continuous bounded functions of t.

The system of differential equations

$$\frac{dy_j}{dt} = \sum_{s=1}^{n} d_{js}(t)\, y_s \qquad (j = 1, \ldots, n) \tag{1.9}$$

will be called the auxiliary system.

In the sequel it will be shown that under suitable conditions, for small values of the parameter $\mu > 0$, the trajectories of the original system of equations (1.1) converge towards the trajectories of the degenerate system (1.5) and that the stability of the auxiliary system (1.9) implies the stability of the original system (1.1).

Theorem 1.1. Suppose that the given system (1.1) and its corresponding degenerate system (1.5) satisfy the following conditions:

1) The system (1.8) is uniformly asymptotically stable;

2) The system of equations with constant coefficients

$$\frac{dy_j}{dt} = \sum_{s=1}^{n} d_{js}(0)\, y_s \qquad (j = 1, \ldots, n)$$

is equi-asymptotically stable for all (fixed) values of θ with $\theta \in [t_0, \infty]$; or, what is the same, the roots of the characteristic equation

$$|d_{js} - \rho\delta_{js}| = 0$$

satisfy the condition

$$\operatorname{Re} \rho < -\Delta, \qquad \Delta = \text{const} > 0 \qquad (\delta_{js} = 0, \text{ if } j \neq s,\ \delta_{js} = 1, \text{ if } j = s)$$

Then for all sufficiently small values of $\mu > 0$ the system (1.1) is uniformly asymptotically stable, and given $Q > 0$ and $\epsilon > 0$ there exists a number $\mu_0 > 0$ such that

$$\begin{aligned} |x_i(t, \mu) - x_i(t)| < \varepsilon, \quad & |y_j(t, \mu) - y_j(t)| < \varepsilon \\ & |y(t_0, \mu) - y_j(t_0)| < Q \quad \text{for } t > t_1(Q, \varepsilon) \end{aligned} \tag{1.10}$$

provided only that $\mu < \mu_0$. The number μ_0 may be chosen so small that the number t_1 appearing in (1.10) will differ from the number t_0 by less than a number $\gamma > 0$ given in advance.

Proof. Let us introduce functions ξ_i and η_j by means of the equations

$$\xi_i(t, \mu) = x_i(t, \mu) - x_i(t), \qquad \eta_j(t, \mu) = y_j(t, \mu) - \sum_{k=1}^{m} \lambda_{jk}(t)\, x_k(t, \mu)$$

and consider the equations of the perturbation of the motion

$$\frac{d\xi_i(t, \mu)}{dt} = \frac{dx_i(t, \mu)}{dt} - \frac{dx_i(t)}{dt}$$

$$\frac{d\eta_j(t, \mu)}{dt} = \frac{dy_j(t, \mu)}{dt} - \sum_{k=1}^{m} \frac{d\lambda_{jk}(t)}{dt} x_k(t, \mu) - \sum_{k=1}^{m} \lambda_{jk}(t) \frac{dx_k(t, \mu)}{dt}$$

In view of Equations (1.1) and (1.8) we obtain

$$\frac{d\xi_i(t, \mu)}{dt} = \sum_{l=1}^{m} r_{il}(t)\, \xi_l(t, \mu) - \sum_{s=1}^{n} b_{is}(t)\, \eta_s(t, \mu)$$

$$\begin{aligned} \frac{d\eta_j(t, \mu)}{dt} = \frac{1}{\mu} \sum_{s=1}^{n} d_{js}(t)\, \eta_s(t, \mu) - \sum_{k=1}^{m} \lambda_{jk}(t) \sum_{l=1}^{m} r_{kl}(t)\, \xi_l(t, \mu) - \\ - \sum_{k=1}^{m} \frac{d\lambda_{jk}(t)}{dt} \xi_k(t, \mu) - \sum_{k=1}^{m} \lambda_{jk}(t) \sum_{l=1}^{m} r_{kl}(t)\, x_l(t) - \\ - \sum_{k=1}^{m} \frac{d\lambda_{jk}(t)}{dt} x_k(t) - \sum_{k=1}^{m} \lambda_{jk}(t) \sum_{s=1}^{m} b_{ks}(t)\, \eta_s(t, \mu) - \sum_{k=1}^{m} \lambda_{jk}(t)\, f_i(t) \end{aligned} \qquad (1.11)$$

In order to prove the theorem it is necessary to prove that the solution $\xi_l(t, \mu)$, $\eta_j(t, \mu)$ of the systems (1.11) satisfies the following condition: given Q, ϵ and δ in advance, there exists a number $\mu_0 > 0$ such that for arbitrary initial data $\xi_{i0} = 0$, $|\eta_{j0}| < Q$ one has as a consequence that the inequalities $|\xi_i(t, \mu)| < \epsilon$, $|\eta_j(t, \mu)| < \epsilon$ hold for $t > t_0 + \delta$, provided that $\mu < \mu_0$.

In view of the conditions of the theorem, the system of linear differential equations with variable coefficients (1.8) is uniformly asymptotically stable. Consequently, there exists a positive-definite quadratic form $v(t, \xi_1, \ldots, \xi_m) = v(t, \xi_i)$ whose total derivative, in view of the system (1.8), is a negative-definite form [5]. Now, the system (1.9) is asymptotically stable for fixed θ, uniformly with respect to θ. Thus for these systems there also exists a positive-definite quadratic form $w(\theta, \eta_1, \ldots, \eta_n) = w(\theta, \eta_j)$ whose total derivative, in view of (1.9), is for θ = const a negative-definite form; and furthermore the partial derivative $\partial w/\partial\theta(\theta, \eta_j)$ is bounded and the positive-definite form $w(\theta, \eta_j)$ and the negative-definite form $\partial w/\partial\theta(\theta, \eta_j)$ relative to the system (1.9) are bounded away from zero uniformly with respect to the parameter θ.

Let us now construct the positive-definite form

$$u(t, \xi_i, \eta_j) = v(t, \xi_i) + w(t, \eta_j) \tag{1.12}$$

and let us show that the total derivative du/dt of this form, computed taking into account the system (1.11), is, for sufficiently small values of the parameter μ, a negative-definite form everywhere outside a small neighborhood of the origin of coordinates $\xi_i = 0$, $\eta_j = 0$.

Indeed

$$\begin{aligned}\frac{du(t, \xi_i, \eta_j)}{dt} &= \frac{\partial v(t, \xi_i)}{\partial t} + \sum_{i=1}^{m} \frac{\partial v(t, \xi_i)}{\partial \xi_i}\left[\sum_{l=1}^{m} r_{il}(t)\xi_l(t, \mu) - \sum_{s=1}^{n} b_{is}(t)\eta_s(t, \mu)\right] + \\ &+ \frac{\partial w(t, \eta_j)}{\partial t} + \sum_{j=1}^{n} \frac{\partial w(t, \eta_j)}{\partial n_j}\left[\frac{1}{\mu}\sum_{s=1}^{n} d_{js}(t)\eta_s(t, \mu) - \right. \\ &- \sum_{k=1}^{m} \lambda_{jk}(t)\sum_{l=1}^{m} r_{kl}(t)\,\xi_l(t,\mu) - \sum_{k=1}^{m}\frac{d\lambda_{ik}(t)}{dt}\xi_k(t, \mu) - \sum_{k=1}^{m}\lambda_{jk}(t)\sum_{l=1}^{m} r_{kl}(t)x_l(t) - \\ &- \left.\sum_{k=1}^{m}\frac{d\lambda_{jk}(t)}{dt}x_k(t) - \sum_{k=1}^{m}\lambda_{jk}(t)\sum_{s=1}^{n} b_{ks}(t)\eta_s(t, \mu) - \sum_{k=1}^{m}\lambda_{jk}(t)f_i(t)\right]\end{aligned} \tag{1.13}$$

Let us note that in the domains $M < |\xi_i|$, $M < |\eta_j|$, where M is a suitably chosen positive number, the sign of $du(t, \xi_i, \eta_j)/dt$ is defined by the sign of the quadratic form in ξ_i and η_j which occurs on the right-hand side of (1.13). Let us now construct the matrix of coefficients of the quadratic form of ξ_i, η_j which is given by $-\, du(t, \xi_i, \eta_j)/dt$, that is, the matrix which is the negative of that occurring on the right-hand side of (1.13):

$$\left\|\begin{matrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} & \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} & \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} & \beta_{m1} & \beta_{m2} & \cdots & \beta_{mn} \\ \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} & \frac{1}{\mu}\delta_{11} + \sigma_{11} & \frac{1}{\mu}\delta_{12} + \sigma_{12} & \cdots & \frac{1}{\mu}\delta_{1n} + \sigma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} & \frac{1}{\mu}\delta_{21} + \sigma_{21} & \frac{1}{\mu}\delta_{22} + \sigma_{22} & \cdots & \frac{1}{\mu}\delta_{2n} + \sigma_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nm} & \frac{1}{\mu}\delta_{n1} + \sigma_{n1} & \frac{1}{\mu}\delta_{n2} + \sigma_{n2} & \cdots & \frac{1}{\mu}\delta_{nn} + \sigma_{nn} \end{matrix}\right\| \tag{1.14}$$

where the coefficients a_{il} are the coefficients in the quadratic form

dv/dt in view of Equations (1.8); the coefficients β_{js} are those of the quadratic form

$$\sum_{i=1}^{m} \frac{\partial v(t, \xi_i)}{\partial \xi_i} \sum_{s=1}^{n} b_{is}(t)\, \eta_s(t, \mu)$$

The expressions $(1/\mu)\delta_{js} + \sigma_{js}$ are the coefficients of the quadratic form

$$-\left\{\frac{\partial w(t, \eta_j)}{\partial t} + \sum_{j=1}^{n} \frac{\partial w(t, \eta_j)}{\partial \eta_j}\left[\frac{1}{\mu}\sum_{s=1}^{n} d_{js}(t)\,\eta_s(t, \mu) - \sum_{k=1}^{m} \lambda_{jk}(t) \sum_{s=1}^{n} b_{ks}(t)\,\eta_s(t, \mu)\right]\right\}$$

Finally, the γ_{jl} are the coefficients of the quadratic form

$$\sum_{j=1}^{n} \frac{\partial w(t, \eta_j)}{\partial \eta_j}\left[\sum_{k=1}^{m} \lambda_{jk}(t) \sum_{l=1}^{m} r_{kl}(t)\, \xi_l(t, \mu) + \sum_{k=1}^{m} \frac{d\lambda_{jk}(t)}{dt}\, \xi_k(t, \mu)\right]$$

In view of Sylvester's criterion for positive quadratic forms, and in view of our hypotheses concerning the functions $v(t, \xi_i)$ and $w(t, \eta_j)$, all the principal diagonal minors of the matrices

$$\left\|\begin{matrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} \end{matrix}\right\| \qquad \left\|\begin{matrix} \frac{1}{\mu}\delta_{11} & \frac{1}{\mu}\delta_{12} & \cdots & \frac{1}{\mu}\delta_{1n} \\ \frac{1}{\mu}\delta_{21} & \frac{1}{\mu}\delta_{22} & \cdots & \frac{1}{\mu}\delta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\mu}\delta_{n1} & \frac{1}{\mu}\delta_{n2} & \cdots & \frac{1}{\mu}\delta_{nn} \end{matrix}\right\|$$

are positive. From this it follows that the principal diagonal minors of the matrix (1.14) are also positive for sufficiently small μ, i.e. the quadratic form used in the construction of $du(t, \xi_i, \eta_j)/dt$ is negative-definite.

Let us show that as μ approaches zero the domain outside of which the total derivative of the form $u(t, \xi_i, \eta_j)$ is negative-definite also approaches zero; that is to say, that as $\mu \to 0$ one has also that $M \to 0$. The expression for $du(t, \xi_i, \eta_j)/dt$ may be written

$$du(t, \xi_i, \eta_j)/dt = K(t, \xi_i, \eta_j) + L(t, \eta_j)$$

where $K(t, \xi_i, \eta_j)$ is a quadratic form in the variables ξ_i, η_j, and $L(t, \eta_j)$ is a linear form in the variables η_j. Let us now add and subtract, to the right-hand side of the last equation, the quadratic form $(\omega/\mu)(\eta_1^2 + \ldots + \eta_n^2)$, where ω is a small positive number. From the known dependence of the coefficients of the quadratic form $K(t, \xi_i, \eta_j)$ on μ, and the smallness (for ω small) of the coefficients of the form $(\omega/\mu)(\eta_1^2 + \ldots + \eta_n^2)$ with respect to the coefficients in η_j of the form

$K(t, \xi_i, \eta_j)$, it follows that the form

$$K_1(t, \xi_i, \eta_j) = K(t, \xi_i, \eta_j) + (\omega/\mu)(\eta_1^2 + \ldots + \eta_n^2)$$

is negative-definite. Consider now the difference

$$L(t, \eta_j) - (\omega/\mu)(\eta_1^2 + \ldots + \eta_n^2)$$

As $\mu \to 0$ the domain $|\eta_j| > M$, outside of which this difference is negative-definite, reduces to the surface $\eta_j = 0$. But from this it follows that the form

$$\frac{du(t, \xi_i, \eta_j)}{dt} = K_1(t, \xi_i, \eta_j) + L(t, \eta_j) - \frac{\omega}{\mu}\sum_{j=1}^{n} \eta_j^2$$

is negative-definite outside an arbitrarily small neighborhood of the origin $\xi_i = 0$, $\eta_j = 0$, provided that the number μ is sufficiently small. Consequently, the functions $u(t, \xi_i, \eta_j)$ and $du(t, \xi_i, \eta_j)/dt$ satisfy the following conditions:

$$u(t, \xi_i, \eta_j) \geqslant \Omega_1(\xi_i, \eta_j), \qquad \frac{du(t, \xi_i, \eta_j)}{dt} \leqslant -\Omega_2(\xi_i, \eta_j) \text{ for } |\xi_i| > M, \ |\xi_j| > M$$

where $\Omega_1(\xi_i, \eta_j)$ and $\Omega_2(\xi_i, \eta_j)$ are positive forms which do not depend upon t.

Taking into account the positive-definiteness of the form $\Omega_1(\xi_i, \eta_j)$ in the space ξ_i, η_j, let us consider a level surface $\Omega_1(\xi_i, \eta_j) = C_1$ such that the moving surface $u(t, \xi_i, \eta_j) = C_1$, containing the initial point of the trajectory of the disturbed motion $P(\xi_{i0}, \eta_{j0})$, lies inside the surface $\Omega_1(\xi_i, \eta_j) = C_1$. In view of this mentioned negative sign of $du(t, \xi_i, \eta_j)/dt$ on this moving surface, we arrive at the conclusion that the trajectory of the system (1.11) remains inside the moving surface $u(t, \xi_i, \eta_j) = C_1$ for all $t \geqslant t_0$.

Further, consider a second level surface $\Omega_1(\xi_i, \eta_j) = C_2$, where $C_2 < C_1$, and such that the moving surface $u(t, \xi_i, \eta_j) = C_2$ contains the domain determined by the number M. If a trajectory of the system (1.11) were outside the domain defined by the level surface $u(t, \xi_i, \eta_j) = C_2$ for all time, then the inequality $du(t, \xi_i, \eta_j)/dt < -\alpha$ would hold for some positive α. From this, in view of the equation

$$u(t, \xi_i, \eta_i) = u(t_0, \xi_{i0}, \eta_{j0}) + \int_{t_0}^{t} \frac{du(t, \xi_i, \eta_j)}{dt}\, dt$$

we obtain

$$u(t, \xi_i, \eta_j) \leqslant u(t_0, \xi_{i0}, \eta_{j0}) - \alpha(t - t_0)$$

which leads to a contradiction. Therefore, the trajectories of the system of equations (1.11) must eventually enter, and remain inside, the domain bounded by a second level surface $u(t, \xi_i, \eta_j) = C_2$.

It remains to show that the quantities η_j decrease rapidly to less than ϵ. For this we shall consider only the trajectories of the system (1.11) which are determined by the initial conditions $\xi_{i0} = 0$, that is, by the conditions $x_i(t_0, \mu) = x_i(t_0)$. Let us show that such a trajectory of (1.11) approaches the origin of coordinates in an arbitrarily small time interval, provided that the parameter μ itself is sufficiently small.

Indeed, the derivative of the form $u(t, \xi_i, \eta_j)$ with respect to time, as was shown earlier, is negative and depends on μ in such a way that the coefficients with respect to $\eta_j\eta_s$ of the quadratic form $k(t, \xi_i, \eta_j)$ contain terms with $1/\mu$ as a factor. Therefore, as μ approaches zero these coefficients increase indefinitely, as does also the absolute value of the negative derivative $du(t, \xi_i, \eta_j)/dt$ when subjected to the condition $|\eta_j| > M$. From this it follows that the function $u(t, \xi_i, \eta_j)$ remains positive-definite, decreases rapidly in absolute value, and that the trajectories of (1.11) approach the origin of coordinates $\xi_i = 0$, $\eta_j = 0$ in an arbitrarily small time interval. (The quantities η_j must decrease rapidly, and at the same time the quantities $\xi_i(t, \mu)$ cannot increase rapidly, in view of the continuity of the integrals.) For all succeeding times the trajectories of the system of equations (1.11) must remain in a small neighborhood of the origin of coordinates. That is, the trajectories of the original system (1.11) approach arbitrarily fast to a sufficiently small neighborhood of the trajectories of the degenerate system of equations (1.5), and remain in this neighborhood for all time to come.

Let us now verify the asymptotic stability of the system (1.1). In order to do this, let us consider two solutions of this system with different initial values, or, what amounts to the same thing, two solutions of (1.11) with different initial conditions. Let us designate these solutions by

$$\alpha_i(t, \mu) = \xi_{i1}(t, \mu) - \xi_{i2}(t, \mu), \qquad \beta_j(t, \mu) = \eta_{j1}(t, \mu) - \eta_{j2}(t, \mu) \quad (1.15)$$

From (1.15) and (1.11) it then follows that

$$\frac{d\alpha_i(t, \mu)}{dt} = \sum_{l=1}^{m} r_{il}(t)\alpha_l(t, \mu) - \sum_{s=1}^{n} b_{is}(t)\beta_s(t, \mu)$$

$$\frac{d\beta_j(t, \mu)}{dt} = \frac{1}{\mu}\sum_{s=1}^{n} d_{js}(t)\beta_s(t, \mu) - \sum_{k=1}^{m}\lambda_{jk}(t)\sum_{l=1}^{m} r_{kl}(t)\alpha_l(t, \mu) -$$
$$- \sum_{k=1}^{m}\frac{d\lambda_{jk}(t)}{dt}\alpha_k(t, \mu) - \sum_{k=1}^{m}\lambda_{jk}\sum_{s=1}^{m} b_{ks}(t)\beta_s(t, \mu) \qquad (1.16)$$

Consider now the positive-definite form

$$u(t, \alpha_i, \beta_j) = v(t, \alpha_i) + w(t, \beta_j) \tag{1.17}$$

which is constructed as in (1.12). This form, for sufficiently small values of the parameter μ, has a negative-definite derivative $du(t, \xi_i, \eta_j)/dt$, which may be computed taking (1.16) into account. This computation proceeds analogously to the discussion above concerning (1.13). Consequently, the form (1.17) is a Liapunov function for the system (1.16), and this guarantees the asymptotic stability of the system (1.16). The asymptotic stability of the system (1.16) then implies the asymptotic stability of the system (1.1).

2. Nonlinear systems. Consider the nonlinear system

$$\begin{aligned} \frac{dx_i}{dt} &= X_i(x_s, y_k, t), \qquad x_{i0} = a_{i0} \qquad (i, s = 1, \ldots, m) \\ \mu \frac{dy_j}{dt} &= Y_j(x_s, y_k, t), \qquad y_{j0} = b_{j0} \qquad (j, k = 1, \ldots, n) \end{aligned} \tag{2.1}$$

where the parameter μ is positive, and the following notation has been employed for the sake of brevity:

$$\begin{aligned} X_i(x_s, y_k, t) &= X_i(x_1, \ldots, x_m, y_1, \ldots, y_n, t) \\ Y_j(x_s, y_k, t) &= Y_j(x_1, \ldots, x_m, y_1, \ldots, y_n, t) \end{aligned}$$

Furthermore, in what follows the range of the indices will always be: $i, s = 1, \ldots, m$ and $j, k = 1, \ldots, n$.

It will be supposed that the functions $X_i(x_s, y_k, t)$ and $Y_j(x_s, y_k, t)$ have continuous bounded derivatives with respect to all their arguments in the domain $|x_s| \leqslant \infty$, $|y_k| \leqslant \infty$, $t_0 \leqslant t < \infty$, and that $D(Y_1, \ldots, Y_n)/D(y_1, \ldots, y_n) \neq 0$.

The degenerate system of differential equations obtained by setting $\mu = 0$ in (2.1) is

$$\frac{dx_i}{dt} = X_i(x_s, y_k, t), \qquad Y_j(x_s, y_k, t) = 0, \qquad x_{i0} = a_{i0} \tag{2.2}$$

It will be supposed that the system of n equations $Y_j(x_s, y_k, t) = 0$ has as a solution the functions

$$y_j = f_j(x_s, t) \qquad (j = 2, \ldots, n) \tag{2.3}$$

whose partial derivatives with respect to x_s and t are bounded.

Let us substitute the functions $y_1, \ldots, y_n$ of (2.3) into the first m

equations of the degenerate system (2.2). We obtain the following system of m equations in the x_i:

$$\frac{dx_i}{dt} = X_i [x_s, f_k, (x_s, t), t] = F_i (x_s, t), \qquad x_{i0} = a_{i0} \tag{2.4}$$

Let the solution of the original system (2.1) corresponding to the given initial conditions be

$$x_i = x_i (t, \mu), \qquad y_j = y_j (t, \mu) \tag{2.5}$$

and the solution of the degenerate system (2.2) with the corresponding initial conditions be

$$x_i = x_i (t), \qquad y_j = y_j (t) = f_j [x_s (t), t] \tag{2.6}$$

Let us construct the auxiliary system of equations of disturbed motion relative to a given solution $x_i = x_i(t)$ of the system (2.4), starting with the equation $z_i = x_i^*(t) - x_i(t)$, where $x_i^*(t)$ is a solution of the system (2.4) which corresponds to a perturbation of the initial conditions $\Delta a_{i0} = a_{i0}^* - a_{i0}$; we obtain

$$\frac{dz_i}{dt} = F_i [z_s + x_s (t), t] - F_i [x_s (t), t]$$
$$= X_i [z_s + x_s (t), f_k (z_s + x_s (t), t), t] - X_i [x_s (t), f_k (x_s (t), t), t] \tag{2.7}$$

It will be supposed that the linear approximation to the system of equations of disturbed motion (2.7) is asymptotically uniformly stable; that is, that the system (2.8) below is uniformly asymptotically stable:

$$\frac{dz_i}{dt} = \sum_{s=1}^{m} g_{is} (t) z_s \qquad \left(g_{is} (t) = \left[\frac{\partial X_i}{\partial x_s} + \sum_{k=1}^{n} \frac{\partial X_i}{\partial y_k}, \quad \frac{\partial f_k}{\partial x_s} \right]_{(x_s = x_s (t))} \right) \tag{2.8}$$

Together with the above systems we shall also consider the system

$$\frac{dy_j}{dt} = Y_j (\alpha_s, y_k, \beta) \tag{2.9}$$

where a_s takes the place of x_s and β takes the place of t. Let us assume that for each fixed set of values $a_s = x_s(\beta)$, $|x_s| < \infty$, $\beta = t$, $t_0 \leqslant \beta < \infty$ there exists a symmetric matrix of constants $A(a_s, \beta)$ with positive eigenvalues ρ_i (which is supposed to be uniformly bounded with respect to a_s and β and to be such that $\rho_i(a_s, \beta) > \Delta > 0$) and such that the symmetrized matrix

$$\{B\}_{jk} = \left(\left\{ A \frac{\partial Y}{\partial y} \right\}_{jk} + \left\{ A \frac{\partial Y}{\partial y} \right\}_{kj}\right) \qquad \left(\left\{\frac{\partial Y}{\partial y}\right\}_{jk} = \frac{\partial Y_j}{\partial y_k}\right) \tag{2.10}$$

has negative eigenvalues r_j which satisfy the inequality $r_i < -\gamma$ (with $\gamma = \text{const} > 0$) for all values $|y_j| \leqslant \infty$. Under these circumstances every solution y_j of the system (2.9) is asymptotically stable [6], regardless of the initial conditions y_{j0}.

Let us construct the system of disturbed motion corresponding to the system (2.1), starting with the equations

$$\xi_i(t, \mu) = x_i(t, \mu) - x_i(t), \qquad \eta_j(t, \mu) = y_j(t, \mu) - f_j[x_s(t, \mu), t]$$

We obtain the system

$$\frac{d\xi_i(t, \mu)}{dt} = \frac{dx_i(t, \mu)}{dt} - \frac{dx_i(t)}{dt}$$

$$\frac{d\eta_j(t, \mu)}{dt} = \frac{dy_j(t, \mu)}{dt} - \sum_{s=1}^{m} \frac{\partial f_j}{\partial x_s} \frac{dx_s(t, \mu)}{dt} - \frac{\partial f_j}{\partial t}$$

whose first m equations are just

$$\frac{d\xi_i(t, \mu)}{dt} = X_i[x_s(t, \mu), y_k(t, \mu), t] - X_i[x_s(t), y_k(t), t]$$

or, equivalently

$$\frac{d\xi_i(t, \mu)}{dt} = X_i[x_s(t) + \xi_s(t, \mu), f_k[x_s(t) + \xi_s(t, \mu), t] + \eta_k(t, \mu), t] -$$
$$- X_i[x_s(t), f_k[x_s(t), t], t] =$$
$$= \sum_{s=1}^{m} \left[\frac{\partial X_i}{\partial x_s} + \sum_{k=1}^{n} \frac{\partial X_i}{\partial y_k} \frac{\partial f_k}{\partial x_s}\right] \xi_s(t, \mu) + \sum_{k=1}^{n} \frac{\partial X_i^*}{\partial y_k} \eta_k(t, \mu) + R_i(\xi_s)$$

In the last equation Taylor's theorem for functions of several variables has been used; the next to the last term represents the increment of the function $X[x_s, y_k, t]$ with respect to $y_k(t)$, and $R_i(\xi_s)$ refers to second and higher order terms with respect to $\xi_s(t, \mu)$. This equation may be abbreviated thus:

$$\frac{d\xi_i(t, \mu)}{dt} = \sum_{s=1}^{m} g_{is}(t)\, \xi_s(t, \mu) + \sum_{k=1}^{n} \frac{\partial X_i^*}{\partial y_k} \eta_k(t, \mu) + R_i(\xi_s)$$

The last n equations of the disturbed motion are

$$\frac{d\eta_j(t, \mu)}{dt} = \frac{dy_j(t, \mu)}{dt} - \sum_{s=1}^{m} \frac{\partial f_j}{\partial x_s} \frac{dx_s(t, \mu)}{dt} - \frac{\partial f_j}{\partial t}$$

or

$$\frac{d\eta_j(t,\mu)}{dt} = \frac{Y_j[x_s(t,\mu), y_k(t,\mu), t]}{\mu} - \sum_{s=1}^{m} \frac{\partial f_j}{\partial x_s} X_s[x_s(t,\mu), y_k(t,\mu), t] -$$

$$- \frac{\partial f_j}{\partial t} = \frac{Y_j[x_s(t)+\xi_s(t,\mu), f_k[x_s(t)+\xi_s(t,\mu), t]+\eta_k(t,\mu), t]}{\mu} -$$

$$- \sum_{s=1}^{m} \frac{\partial f_j}{\partial x_s} X_s[x_s(t)+\xi_s(t,\mu), f_k[x_s(t)+\xi_s(t,\mu), t]+\eta_k(t,\mu), t] -$$

$$- \sum_{s=1}^{m} \frac{\partial f_j}{\partial x_s} X_s[x_s(t), f_k[x_s(t), t], t] - \frac{\partial f_j}{\partial t} + \sum_{s=1}^{m} \frac{\partial f_j}{\partial x_s} X_s[x_s(t), f_k[x_s(t), t], t]$$

For brevity, let us write $x(t) + \xi_s(t, \mu) = a^*(t)$; then this system of n equations may be rewritten

$$\frac{d\eta_j(t,\mu)}{dt} = \frac{Y_j[\alpha_s^*(t), f_k[\alpha_s^*(t), t]+\eta_k(t,\mu), t]}{\mu} + \sum_{l=1}^{m} \gamma_{jl}(t)\xi_l(t,\mu) +$$

$$+ \sum_{k=1}^{n} \gamma_{jk}^*\eta_k(t,\mu) + R_j^*(\xi_s) + Q_j(t) \qquad \left(\gamma_{jl}(t) = \sum_{s=1}^{m} \frac{\partial f_j}{\partial x_s} g_{sl}(t)\right)$$

where the terms $\gamma_{j1}^*\eta_1(t, \mu) + \ldots + \gamma_{jn}^* \eta_n(t, \mu)$ take the place of the sums of the derivatives df_j/dx_l when the increments of the functions $X_s[x_s(t), y_k(t), t]$ with respect to $y_k(t)$ are written by means of Taylor's theorem. The symbol $R_j^*(\xi_s)$ stands for a collection of terms which contain the functions $\xi_s(t, \mu)$ to the second or higher orders; and, finally, $Q(t)$ is a function of t.

Thus, the system of equations of the disturbed motion has the form

$$\frac{d\xi_i(t,\mu)}{dt} = \sum_{s=1}^{m} g_{is}(t)\xi_s(t,\mu) + \sum_{k=1}^{n} \frac{\partial X_i^*}{\partial y_k}\eta_k(t,\mu) + R_i(\xi_s) \tag{2.11}$$

$$\frac{d\eta_j(t,\mu)}{dt} = \frac{1}{\mu} Y_j\{\alpha_s^*(t), f_k[\alpha_s^*(t), t] + \eta_k(t,\mu), t\} + \sum_{l=1}^{m} \gamma_{jl}(t)\xi_l(t,\mu) +$$

$$+ \sum_{k=1}^{n} \gamma_{jk}^*\eta_k(t,\mu) + R_j^*(\xi_s) + Q_j(t) \qquad (Y_j\{\alpha_s^*, f_k[\alpha_s^*, t], t\} = 0)$$

By hypothesis, the system (2.8) is uniformly asymptotically stable. Consequently, there exists a positive-definite quadratic form $v(t, \xi_s) = v(t, \xi_1, \ldots, \xi_m)$ whose total derivative, calculated making use of the system of equations (2.8), is a negative-definite quadratic form. But it was also supposed above that the system of equations (2.9) has the property that for each fixed set of values $a_s = x_s(t) = x_s(\beta)$ and $\beta = t$, $t_0 \leqslant \beta < \infty$ there exists a symmetric matrix $A(a_s, \beta)$ having positive eigenvalues and satisfying Equation (2.10). Therefore, the system of equations (2.9) possesses a Liapunov function $w(a_s, \beta, \eta_k) = w(a_1, \ldots, a_m, \beta, \eta_1, \ldots, \eta_n)$, $\eta_k = y_k - f_k(a_s, \beta)$ whose total derivative, computed taking the system (2.9) into account, satisfies the inequalities (16.22), (16.23) and (16.24) which appear in [6].

Let us construct the positive function

$$u(t, \xi_i, \eta_j) = v(t, \xi_i) + w[x_s(t), t, \eta_j] \tag{2.12}$$

and let us prove that the total derivative of this function, $du(t, \xi_i, \eta_j)/dt$, computed taking into account the perturbed system of equations (2.11), is a negative-definite function when the parameter μ is sufficiently small. Indeed, the total derivative of the function $u(t, \xi_i, \eta_j)$, the system (2.11) being taken into account, is just

$$\frac{du(t, \xi_i, \eta_j)}{dt} = \frac{\partial v(t, \xi_i)}{\partial t} + \sum_{i=1}^{m} \frac{\partial v(t, \xi_i)}{\partial \xi_i}\Big[\sum_{s=1}^{m} g_{is}(t)\xi_s(t, \mu) + \sum_{k=1}^{n} \frac{\partial X_i^*}{\partial y_k}\eta_k(t, \mu) + R_i(\xi_s)\Big] + \sum_{s=1}^{m} \frac{\partial w[x_s(t), t, \eta_j]}{\partial x_s}\frac{dx_s}{dt} + \frac{\partial w[x_s(t), t, \eta_j]}{\partial t} +$$
$$+ \sum_{j=1}^{n} \frac{\partial w[x_s(t), t, \eta_j]}{\partial \eta_j}\Big[\frac{1}{\mu}Y_j(\alpha_s^*(t), f_k[\alpha_s^*(t), t] + \eta_k(t, \mu), t) + \sum_{l=1}^{m} \gamma_{jl}(t)\xi_l(t, \mu) + \sum_{k=1}^{n} \gamma_{jk}^*\eta_k(t, \mu) + R_j^*(\xi_s) + Q_j(t)\Big] \tag{2.13}$$

Since the function $w[x_s(t), t, \eta_j]$ satisfies inequalities analogous to those for quadratic forms (see [6]), under the hypotheses that the functions $X_i[x_s(t), y_k(t), t]$ and $Y_j[x_s(t), y_k(t), t]$ have bounded partial derivatives, we are led to the following inequality:

$$\sum_{s=1}^{m} \frac{\partial w[x_s(t), t, \eta_j]}{\partial x_s}\frac{dx_s}{dt} + \frac{\partial w[x_s(t), t, \eta_j]}{\partial t} \leqslant \sum_{k, j=1}^{n} d_{kj}\eta_k\eta_j$$

as long as the $x_i(t, \mu)$ and $y_j(t, \mu)$ remain in a bounded portion of the space $x_i y_j$.

Considering the structure of the derivative $du(t, \xi_i, \eta_j)/dt$ in view of the system of the perturbed motion (2.11), we observe that in spite of its more complicated appearance than in the linear case we can still carry out the considerations leading to bounds for $du(t, \xi_i, \eta_j)/dt$ along the same general lines, as was done before for the linear case (see the discussion following Equation (1.12) above); because, although the functions involved in the present argument need not be quadratic forms, as in the linear case, they still satisfy inequalities which are similar to those satisfied by quadratic forms. Thus we are again led to conclude that the total derivative $du(t, \xi_i, \eta_j)/dt$ is negative-definite outside the domain $|\xi_i| > M$, $|\eta_j| > M$ for sufficiently small values of the parameter $\mu > 0$. Further, as in the linear case, it may be shown that the domain $|\xi_i| > M$, $|\eta_j| > M$, where $M > 0$, approaches zero as $\mu \to 0$, and that the following theorem holds:

Theorem 2.1. Suppose that the system of differential equations containing a small parameter, system (2.1), satisfies the following conditions:

1) the system of equations (2.8), which is the linear approximation to the system of disturbed motions relative to the degenerate system (2.4), is uniformly asymptotically stable;

2) for each set of fixed values a_s and β the system (2.9) is such that there exists a symmetric matrix $A(a_s, \beta)$, uniformly bounded with respect to a_s and β, having the property that the symmetrized matrix $\{B\}_{jk}$ (2.10) has negative eigenvalues satisfying the inequalities $r_i < -\gamma (\gamma = \text{const} > 0)$.

Then for sufficiently small values of the parameter μ the solution of (2.1), $x_i(t, \mu)$, $y_j(t, \mu)$, with $x_i(t_0, \mu) = a_{i0}$, $y_j(t_0, \mu) = b_{j0}$, is uniformly asymptotically stable with respect to small variations of the x_{i0} and arbitrary variations of the y_{j0}. Given $Q > 0$, $\epsilon > 0$, there exists a number $\mu_0 > 0$ such that the following inequality holds:

$$|x_i(t,\mu) - x_i(t)| < \varepsilon, \qquad |y_j(t,\mu) - y_j(t)| < \varepsilon, \qquad \text{for } t > t_1(Q, \varepsilon),$$
$$|y_j(t_0,\mu) - y_j(t_0)| < Q \qquad (2.14)$$

provided only that $\mu < \mu_0$. The number μ_0 may be chosen so small that the number t_0 appearing in (2.14) differs from the number t_1 by less than any preassigned positive number.

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